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Blunt-body impact on the free surface of a compressible liquid

By ALEXANDER KOROBKIN
Lavrentyev Institute of Hydrodynamics, 630090 Novosibirsk, Russia

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The present paper deals with the plane unsteady problem of the penetration of a blunt solid contour into an ideal compressible liquid. At the initial instant of time, the solid body touches the liquid free boundary at a single point. At the initial stage, the duration of which depends on the body geometry, the displacements of liquid particles are small and the disturbed fluid motion is described within the framework of the acoustic approximation. The main feature of the problem is the existence of a contact line between the free surface of the liquid and the solid-body surface. The position of this line is not known in advance and is to be determined together with the solution of the problem. A brief description of the method that provides a solution of complicated nonlinear problems such as this is given. The pressure distribution and the velocity field in the liquid are shown to be given in quadratures and in the case of a parabolic entering contour in an explicit form. For a parabolic entering contour the pressure at the top of the contour calculated using the model of an incompressible liquid is observed to deviate from a precise value by not more than 10% of the latter after the first expansion waves have passed the contact point. The solution analysis enabled us to distinguish the regions in which the acoustic approximation fails and the liquid flow becomes essentially nonlinear.

1. Introduction

The plane unsteady problem of a smooth blunt solid body penetrating an ideal and weakly compressible liquid is considered. Initially the liquid is at rest and occupies a lower half-plane ($y' < 0$), and the body touches its free boundary ($y' = 0$) at a single point (figure 1*a*) taken as the origin of the Cartesian coordinate system $x'Oy'$ (dimensional variables are denoted by a prime). At some instant of time, taken as initial ($t' = 0$), body begins to penetrate the liquid vertically. The velocity of the body V is assumed to be much less than the sound velocity in the resting liquid c_0. External mass forces and surface tension are neglected. It is necessary to determine the liquid flow, its boundary geometry and the pressure distribution along the wetted part of the body for all times of motion.

This problem belongs to a broad class of problems on the unsteady motion of a liquid occupying a domain that changes with time when the fluid boundary consists of the free surface with a touching solid surface and the contact lines between them (e.g. the problem of ship motion, the liquid flow along a dry channel, etc.). It should be noted that in such problems both the liquid flow and the contact line position are to be determined simultaneously at every instant. Thus it is hoped that a fundamental understanding of the main characteristics of the liquid flow initiated by the entry of a solid body into it will be helpful in analysing many related problems. On the other

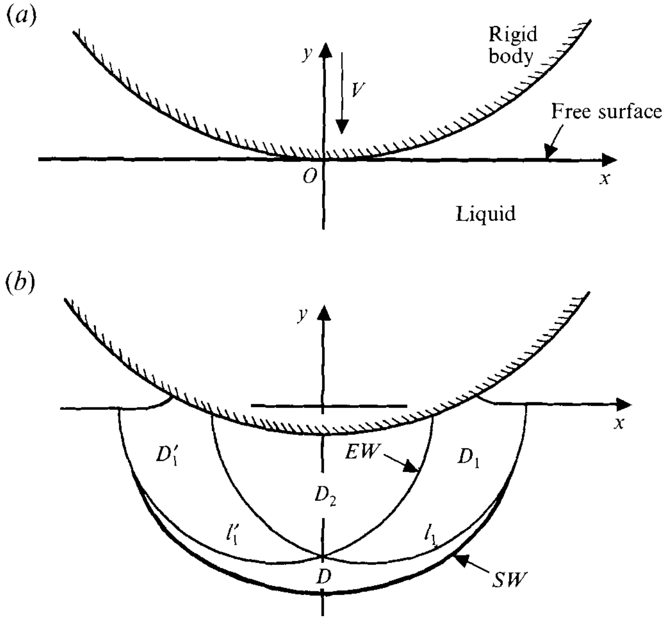

FIGURE 1. Impact by a blunt body on a liquid free surface. (*a*) Initially, liquid is at rest and occupies a lower half-plane $y < 0$, and a body touches the free boundary at a single point. (*b*) The flow pattern at the subsonic stage: SW, shock wave; EW, expansion wave produced by the shock reflection in the free surface; l_1, l_1', the right-hand and the left-hand expansion wave fronts respectively; D_1, D_1', D_2 are the regions under consideration in the present paper.

hand, the entry problem has a distinguishing feature compared to the other problems of this class: namely, initially the whole of the liquid boundary is free and then at some instant of time, a previously absent component of the liquid boundary adjacent to the solid body appears, i.e. at the initial moment the flow topology changes.

A precise statement of the entry problem is very complicated and an accurate solution or some properties of the solution are almost unavailable in practice. Therefore, various simplifications of the liquid properties and its flow structure are used. In most works devoted to the entry problem the liquid is assumed to be incompressible and ideal. For pointed bodies, such as a wedge or a cone, the pressure distribution along the wetted part of the entering body, calculated within the framework of the ideal incompressible liquid model, is in good agreement with the experimental data. For blunt bodies (a sphere, a parabolic shape and so on) this model gives an infinite value of the pressure at the instant of impact, however small the impact velocity may be. This is associated with the fact that the incompressible liquid model, in which the velocity of propagation of disturbances is infinite, is unable to describe the important stage of the entry process for a blunt body. The problem is that just after the moment of the first contact of the liquid with the solid blunt body, the area of the wetted part of the body surface expands at a rate that is larger than the local sound velocity (Bowden & Field 1964). In this case, the disturbance front is attached obliquely to the contact line and the disturbed part of the liquid is bounded by the solid-body surface on one side and by the shock wave on the other side. At this stage, referred to as supersonic, the free boundary of the liquid remains undisturbed and looks like a rigid undeformable plate. To obtain realistic results for this stage of the process, the liquid compressibility must be taken into account independently of value of the Mach number $M = V/c_0$. Later on, the shock front breaks away from the contact line and escapes onto the free surface. For large times, compared with the duration of the supersonic stage, when the shock front has travelled quite far from the contact spot, the initial asymptotics of the solution of the entry problem for the incompressible liquid model well describes liquid motion near the body and makes it possible to start the numerical analysis of the nonlinear effects and real deformations of the liquid boundaries.

The duration of the supersonic stage T' is the important characteristic of the entry process and it depends on both the body geometry and the Mach number of the problem. Its asymptotics at $M \to 0$ can be easy obtained using the condition that the velocities of the contact point at this moment and the sound c_0 are equal (see Bowden & Field 1964). For example, for a parabolic contour one has $T' = \frac{1}{2}(R/V)M^2[1+O(M^{\frac{2}{3}})]$ (see Korobkin & Pukhnachov 1985), where R is the parabola curvature radius at the apex. The significance of the quantity T' is that the liquid compressibility at $t'/T' = O(1)$ cannot be neglected without losing the conformity of the model with actually observed phenomena. If $T'/t' = o(1)$, then the liquid may be considered as incompressible throughout the flow region except for in a small vicinity of the shock front, the ratio of this vicinity width to the distance between the shock front and the body surface tending to zero at $M \to 0$.

To discuss the initial stage of the blunt-body impact on the free surface of a liquid, where the liquid compressibility is the governing factor, let us change to dimensionless variables, which are designated by the above-mentioned terms without a prime. We take the half-width of the wetted part of the body at the end of the supersonic stage L as the lengthscale and the body velocity V as the velocity scale of liquid particles. The quantity L/c_0, being of order identical with T' at $M \to 0$, is conveniently taken as the timescale T_c. In particular, for the parabolic contour $T_0 = 2T'$ and $L = RM$. At the impact stage concerned here the liquid-particle displacements are infinitesimal and are $O(LM)$, which allows us, as a first approximation, to put boundary conditions on the undisturbed initial level of the liquid and to linearize them and the equations of motion near the initial rest state.

The linearization leads to the well-known acoustic approximation where the liquid motion is irrotational and is described by the velocity potential $\phi(x, y, t)$. The potential $\phi(x, y, t)$ satisfies the wave equation in the lower half-plane ($y < 0$), the mixed boundary conditions on the line $y = 0$ and is identically zero at $t < 0$. The characteristic feature of this problem is the fact that the division of the liquid boundary into a free surface and a contact spot is unknown and must be determined with the help of an additional condition. The presence of such a condition renders the problem nonlinear despite the fact that both the equations of motion and the boundary conditions are linearized.

The problem stated in this way has been numerically solved by Gavrilenko (1984) and Gavrilenko & Kubenko (1985) using an iteration method. In this method the size of the contact spot is selected at every time step from the condition that moving liquid particles do not penetrate the 'forbidden' region bounded by the surface of the entering body. In effect this method is the numerical realization of the additional condition suggested by Wagner (1932). The Wagner approach makes it possible to solve analytically the linearized penetration problem within the framework of the incompressible liquid model. Nevertheless, as for the acoustic approximation, the Wagner condition leads to a complicated integral equation, the closed solution of which fails.

That is why a new approach has been proposed (see Korobkin 1990), which is based on the regularization of the initial-value problem for the velocity potential ϕ. Namely, dealing with a new unknown function $\psi(x, y, t)$ is suggested such that the $\psi_t = \phi$ and $\psi(x, y, 0) = 0$. The initial-value problem for the function ψ resembles that of the potential ϕ except for the slip condition on the contact spot, the right-hand side of which now depends on the entering body shape. The introduction of this transformation is based on some ideas suggested by Pukhnachov (1979), and it was first used by Korobkin (1982) to prove the correctness of the mathematical formulation of the entry problem within the incompressible liquid model. Then the transformation was investigated by Ockendon (1990). According to its definition, the function $\psi(x, y, t)$ is a displacement potential and its derivatives ψ_x, ψ_y are the vector components of the liquid particle displacements in the horizontal and vertical directions, respectively. It would appear reasonable to demand that the liquid particle displacements would be finite at every finite instant of time. This simple condition proved to be sufficient for determining the law of motion of the contact points. In this case the function $\psi(x, y, t)$ is continuously differentiable over the flow region and its second derivatives with respect to the space variables are square integrable. This approach has made possible the decomposition of the original nonlinear problem into two parts. First, the initial-value problem for the displacement potential ψ is solved and the law of motion of contact points determined. Thereafter the problem for the potential ϕ becomes linear, as the points where the boundary condition changes its type are now known, which allows us to give the solution in quadratures. At first glance it would seem that the problem for the potential $\psi(x, y, t)$ is no simpler than the original one. However, this is not true, mainly because of the improved smoothness of the unknown function. To solve the original problem completely, the displacement potential itself need not be determined but only its normal derivative on the boundary $\psi_y(x, 0, t)$, i.e. the vertical displacement of liquid particles located on the free surface at the initial instant of time. This function may be found in an explicit form for an arbitrary law of motion of contact points. The condition that the function $\psi_y(x, 0, t)$ is finite leads to a single nonlinear algebraic equation that is used to calculate the time dependence of the position of each contact point.

It is important that the free surface of the liquid remains undisturbed during the supersonic stage ($0 < t < T$) and, hence, the contact points coincide with the points of intersection of the half-plane boundary ($y = 0$) with the entering contour. At this stage there is an analogy between the impact problem and the well-known problem of a plane lifting surface placed in a supersonic stream at a small angle of attack. This analogy was first indicated by Skalak & Feit (1966) and used by Rochester (1979) while analysing the related plane problem of a drop impact on a solid surface. However, at this stage the impact problem may be analysed directly without using this analogy. This has been performed by Korobkin (1992*b*) and the solution has been written in quadratures which makes possible the treatment in detail of the influence of the body geometry on the fluid motion characteristics. In particular, a shallow depression at the body apex is shown to lead to the focusing of the shock wave formed under the high-speed impact. This result correlates with the experimental data by Dear & Field (1988). As a non-trivial consequence of the theoretical analysis it was concluded that the maximum strength of the shock wave is possible only when the depression is a hyperbola in shape.

After escaping the shock wave on the free boundary (the subsonic stage) expansion waves are formed and spread along the contact spot and into the liquid bulk. Then they escape to the free surface again and are reflected from it as compression waves. Such reflections proceed repeatedly, the wave amplitudes reducing with time. When analysing the liquid flow at this stage, the analogy of the impact problem and that of the supersonic flow about a thin wing is very helpful and makes possible an efficient problem solution. This analogy was used by Korobkin (1985) in studying the subsonic stage. In particular, after the shock wave escapes ($t > T$), the pressure at the escape point was shown to decrease and to be of $O(1/\{t-T\}^{\frac{1}{2}})$ at $t \to T+0$. However, the contact-point positions were presumed to be known, and so the results were conditional and therefore unpublished.

It should be noted that the proposed method of solving the blunt-body impact problem differs from the well-known approaches for analysing the impact by a body with flattening or sharpening. The main feature of this method is the availability of the supersonic stage for blunt bodies, which leads to a variety of peculiar phenomena.

The present research should be considered as an extension of the previous study by the author (see Korobkin 1992*b*); all notations are retained. Particular attention is given to the peculiarities of the liquid flow and the pressure distribution, immediately after the escape of the shock wave on the free surface. As mentioned earlier, at every instant a region $D(t)$ (see figure 1*b*) may be indicated in the flow where the liquid motion is independent of the presence of the free boundary and coincides with the corresponding flow in the problem of a contour emerging from an infinite plate. The geometry of region $D(t)$ and the liquid flow inside it has been analysed by Korobkin (1992*b*). After the escape of the shock wave onto the free boundary, regions $D_1(t)$ and $D_1'(t)$ are formed, where the liquid flow essentially depends on the presence of the free surface. The lower boundaries of these regions in the acoustics approximation are half-circles and they are thereafter referred to as fronts of the right-hand and left-hand expansion waves because after they have passed the liquid pressure drops. The calculation of the pressure field behind the wave fronts as well as in the region of their interaction, $D_2(t)$, is of evident interest, since there exist possible cases where these waves cause the appearance of negative pressure zones which may account for cavitation and, as a result, for the cavitation erosion of a body. In addition, such calculations performed for a sufficiently long time may provide estimates of the duration of the initial stage after which the liquid compressibility can be neglected for describing the flow near the entering body.

The entry of a shape that is symmetric about the Oy-axis is considered for simplicity only: all the results except as otherwise noted are also valid in the general case.

2. Formulation of the problem

Within the framework of the acoustic approximation the flow domain coincides with the half-plane $y < 0$ which is occupied by the liquid at the initial moment. The liquid flow is described by the velocity potential $\phi(x, y, t)$, for which the initial boundary-value problem has the form

$$\left.\begin{aligned} \phi_{tt} &= \phi_{xx}+\phi_{yy} \quad (y<0), \\ \phi &= 0 \quad (y=0, |x|>a(t)), \\ \phi_y &= -1 \quad (y=0, |x|<a(t)), \\ \phi &= \phi_t = 0 \quad (y<0, t=0), \\ \phi &\to 0 \quad (y^2+x^2 \to \infty). \end{aligned}\right\} \tag{1}$$

With the choice of characteristic scales of the variations of the independent variables and the unknown functions given in §1, the sound velocity in the liquid at rest and the impact velocity are equal to unity in the non-dimensional variables. The part of the boundary $-a(t) < x < a(t)$, $y = 0$ corresponds to the contact region of the entering contour with the liquid, and the parts $x < -a(t)$ and $x > a(t)$ correspond to the free surface where the pressure is zero at all times. The points with coordinates $x = +a(t)$, $y = 0$ correspond to the contact points of the free liquid boundary with the surface of the rigid body. The function $a(t)$ describes the law of motion of the points and it is assumed to be determined in advance. In the general case, the position of the body contour in non-dimensional variables is given by the equation $y = M(f(x)-t)$, where the function $f(x)$ describes the contour shape, so that at the supersonic stage $f[a(t)] \equiv t$. For a parabolic contour, when $f(x) = \frac{1}{2}x^2$, we have $a(t) = (2t)^{\frac{1}{2}}$ when $0 < t < \frac{1}{2}$ (supersonic stage) and $a(t) = \{3[5+8t]^{\frac{1}{2}}-5\}/4$ (see Korobkin 1992*a*) when $\frac{1}{2} < t < \frac{11}{2}$ (the beginning of the subsonic stage). The time $\frac{11}{2}$ corresponds to the moment when the left-hand expansion wave overtakes the right-hand contact point. Calculations of the function $a(t)$ for large times and also for an arbitrary contour shape may only be done numerically. The calculation length is doubled if the entering body is unsymmetrical: in this case the laws of motion of the right-hand and left-hand contact points have to be determined separately. From the definition of the dimensionless variables one has $a(T) = 1$ in the symmetrical case, but for the possible use of the present results for the problem of an unsymmetrical entering body also, we will denote $a(T)$ by a_*, as was done earlier. When the solution of problem (1) has been found, the pressure in the liquid $p(x, y, t)$ is determined by the linearized Cauchy–Lagrange integral $p = -\phi_t$, and the velocity field $\boldsymbol{u}(x, y, t) = (u, v)$ is given by $\boldsymbol{u} = (\phi_x, \phi_y)$.

As the function $a(t)$ is known, the impact problem (1) is equivalent to the problem of a plane lifting surface placed in a supersonic stream at a small angle of attack, when the stream velocity is the double the sound velocity and the attack angle is half the Mach number M. This last problem was investigated in detail by Krasilshchikova (1954). The theory developed by Krasilshchikova makes it possible to reduce the calculations, but, where possible, the analytical calculations will be done directly (see Korobkin 1992*b*), which leads more rapidly to the final formulae that permit a detailed analysis.

The method of solution is based on the well-known formula

$$\phi(x, y, t) = \frac{1}{\pi}\iint_\sigma \frac{\phi_y(\xi, 0, \tau)\, d\xi\, d\pi}{[(t-\tau)^2-(x-\xi)^2-y^2]^{\frac{1}{2}}}, \tag{2}$$

which makes it possible to evaluate the potential ϕ in the lower half-plane when the vertical component of the velocity vector $v(x, 0, t) = \phi_y(x, 0, t)$ over its boundary and for all time is given. The integration domain $\sigma(x, y, t)$ lies in the upper half-plane $\tau > 0$ and is bounded above by the hyperbola (see figure 2)

$$\Gamma_2: \quad \tau = t-[(x-\xi)^2+y^2]^{\frac{1}{2}} =: F(\xi, x, y, t).$$

When using the formula (2) it is necessary to divide the domain $\sigma(x, y, t)$ into parts, inside each of which the function $v(\xi, 0, \tau)$ is determined by an indivisible analytical expression. For instance, the curve $\Gamma_c: \xi = \pm a(\tau)$ indicates the width of the contact spot at the moment τ (see figure 2). Inside the domain bounded by this curve we have $v(\xi, 0, \tau) \equiv -1$. In the domain external with respect to the curve

$$\Gamma_1: \quad \tau = \begin{cases} f(\xi), & |\xi| < a_*, \\ |\xi|+T-a_*, & |\xi| > a_* \end{cases}$$

the free surface is undisturbed for all time of motion and, hence, $v(\xi, 0, \tau) \equiv 0$. Inside

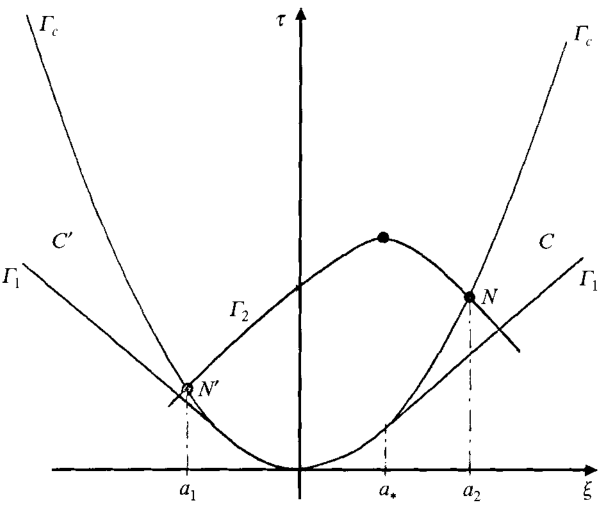

FIGURE 2. Geometry of the integration domain.

the right-hand C region and the left-hand C' region bounded by the curves Γ_c and Γ_1, the function $v(\xi, 0, \tau)$ is unknown. The technique of its determination is based on the theory of thin wings in a supersonic flow (see Krasilshchikova 1982).

When $y = 0$ and $(x, t) \in C$ the left-hand side of (2) is equal to zero in accordance with the boundary condition on the free surface and we get an integral equation with respect to $v(\xi, 0, \tau)$. In the characteristic coordinate system which is produced from the system $\xi O \tau$ by its rotation anticlockwise through 45° the integral equation can be split into two Abel's equations, solutions of which are given by the explicit formulae. Let some point S lie in the region C and its coordinates be x, t. Then at the moment t the vertical velocity of a liquid particle lying at a distance x from the origin of the coordinate system is determined by the curvilinear integral of the first kind

$$v(S) = -\frac{1}{\pi |SK|^{\frac{1}{2}}} \int_{(AK)} v(E) \frac{|KE|^{\frac{1}{2}}}{|SE|} \mathrm{d}s. \tag{3}$$

Here $|SK|$, $|KE|$, $|SE|$ are the lengths of the segments SK, KE, BE respectively (see figure 3). The integration is carried out over the part AK of the line passing through the point S inclined at 45° to the axis $O\xi$ (it is a line with the characteristic slope). The same formula is valid also in the case when the point S lies in the region C': the corresponding picture explaining the notation is produced by mirror reflection of figure 3 with respect to the axis $O\tau$. One needs to consider (3) as a recurrent formula, consistently determining the function $v(\xi, 0, \tau)$ in regions C and C'. As the first step, consider the case when the point S lie sufficiently close to the curve Γ_1 and the segment AN is absent. In this case $v(E) \equiv -1$ over the integration interval and we get the explicit formula

$$v(S) = \frac{2}{\pi}(\zeta - \arctan \zeta), \quad \zeta = \{|AK|/|SK|\}^{\frac{1}{2}}, \tag{4}$$

which is valid in the strips C_1, C_1' adjacent to Γ_1. The width of the strips is equal to $\sqrt{2}$ for symmetrical body shapes. The strip C_2 is considered at the next step. This strip is adjacent to C_1 and it has a width such that under the integration in (3) one has $0 <$

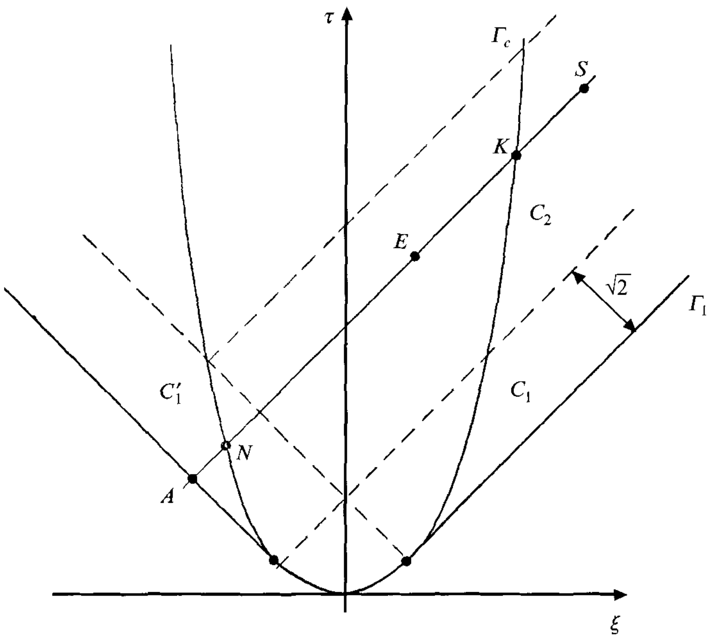

FIGURE 3. Integration path for the calculation of the vertical velocity component of the free surface.

$|AN| < \sqrt{2}$. Then the strip C_2' is considered and so on. For an unsymmetrical shape (4) is also valid but the sizes of the strips in regions C and C' are determined in a more complicated way.

Following this scheme we can determine the function $v(x, 0, t)$ and then use (2) to evaluate the velocity field inside the liquid. Equation (3) is very helpful and makes it possible to reduce the calculations to explicit formulae in some simple cases.

The coordinate of the left-hand intersection point of curves Γ_c and Γ_2 is denoted by a_1 and that of the right-hand point by a_2 (see figure 2). For values of x, y, t such that $-a_* \leqslant a_1 \leqslant a_2 \leqslant a_*$ we must take $v(\xi, 0, \tau) \equiv -1$ in (2). This case corresponds to liquid flow in the region $D(t)$ and was considered in detail by Korobkin (1992*b*). In the present paper situations will be considered when the chain of inequalities given above fails. In §3 the pressure distribution inside region $D_1(t)$ behind the front of the right expansion wave is analysed (see figure 1*b*). Region $D_1'(t)$ is considered in a similar way. In §4 the interaction of the expansion waves is analysed and the pressure inside region $D_2(t)$ is evaluated. Thus the solution of the original problem (1) for $t > 0$ is presented behind the shock front in a domain which is a combination of regions $D(t)$, $D_1(t)$, $D_1'(t)$, $D_2(t)$ (see figure 4). This domain is bounded above by the fronts of the compression waves, which are formed at the moments when the corresponding expansion waves overtake the contact points opposite them and escape onto the free surface of the liquid.

It is worth noting that we need not only to construct the solution of (1), but also to verify its correspondence to the main assumptions which guarantees the validity of the acoustic theory. Leaving aside detailed discussion of the question, we can point out that if the pressure $p(x, y, t)$, the velocities $u(x, y, t)$, $v(x, y, t)$ of the liquid particles and also their first derivatives with respect to the spatial variables are finite then the solution of the problem (1) gives formally the asymptotics of the solution of the

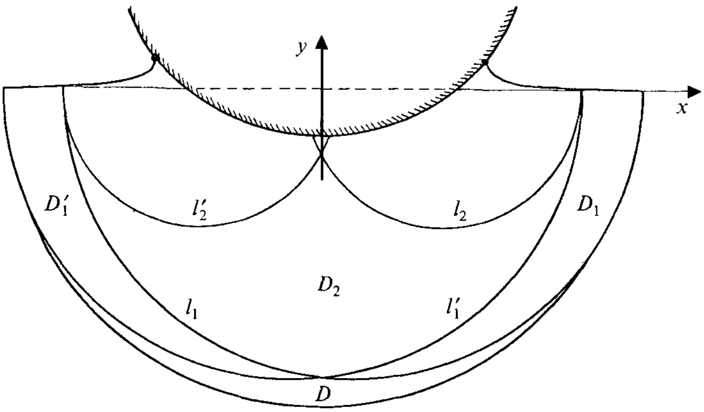

FIGURE 4. The flow pattern at the subsonic stage: l_1, l_1' are the right-hand and the left-hand expansion wave fronts respectively; l_2, l_2' are the right-hand and the left-hand compression wave fronts respectively; D, D', D are the regions considered.

original nonlinear problem of blunt-body impact on a compressible liquid surface as $M \to 0$. The narrow zones where this condition fails must be distinguished and the liquid motion inside those zones has to be analysed separately. The results of this analysis will be published in the near future.

3. The pressure distribution inside region $D_1(t)$

Region $D_1(t)$ appears at the moment of escape of the shock wave onto the free surface ($t = T$). It is bounded below by the front of the right-hand expansion wave which is the circle segment l_1 with its centre at the point $(a_*, 0)$ and radius $t - T$ ($t > T$), and above by the liquid boundary ($y = 0$) and, when $t > a_* + T$, by the front of the left-hand expansion wave which is the circle segment l_1' with its centre at the point $(-a_*, 0)$ and radius $t - T$. Let some point U with coordinates x, y belong to $D_1(t)$ at the moment t, then the following inequalities occur: $|a_1(x,y,t)| \leqslant a_*$, $a_2(x,y,t) \geqslant a_*$; moreover if $U \in l_1$ then $a_2(x,y,t) \equiv a_*$, if $U \in l_1'$ then $a_1(x,y,t) \equiv -a_*$.

In the case under consideration the integration in (2) is carried out over the domain that can be represented as the sum of three domains: $N'PNL$, NAL, NKA (see figure 5). The line NL passes through point N with coordinates a_2, τ_2 where τ_2 is such that $a(\tau_2) = a_2$, and has the characteristic slope. It can be directly verified that the integrals over NAL, NKA differ only in sign (see Krasilshchikova 1982). To demonstrate this we need to rewrite the integral over the domain NKA in the characteristic coordinate system and take into account (3). Therefore the integrals over the domains NAL, NKA mutually vanish and only the integration over $N'PNL$ remains, where $v(\xi, 0, \tau) \equiv -1$. In consequence, for the velocity potential $\phi(x,y,t)$ in $D_1(t)$ we find

$$\phi(x,y,t) = \frac{1}{\pi}\int_{a_1}^{a_2}\left(\int_{H(\xi)}^{F(\xi)} \frac{\mathrm{d}\tau}{[(t-\tau)^2-(x-\xi)^2-y^2]^{\frac{1}{2}}}\right)\mathrm{d}\xi. \tag{5}$$

Here $H(\xi) = f(\xi)$ when $a_1 \leqslant \xi \leqslant \xi_L(x,y,t)$ and $H(\xi) = \xi - a_2 + \tau_2$ when $\xi_L \leqslant \xi \leqslant a_2$, the coordinates of the point L are $(\xi_L, f(\xi_L))$, and the function $\xi_L(x,y,t)$ satisfies the equation $f(\xi_L) = \xi_L - a_2 + \tau_2$. It should be noted that in (5) the differentiation with respect to x, y or t and the integration with respect to ξ are interchangeable, because $F(a_j) = H(a_j)$ for $j = 1, 2$. Evaluating the internal integral in (5) and differentiating both

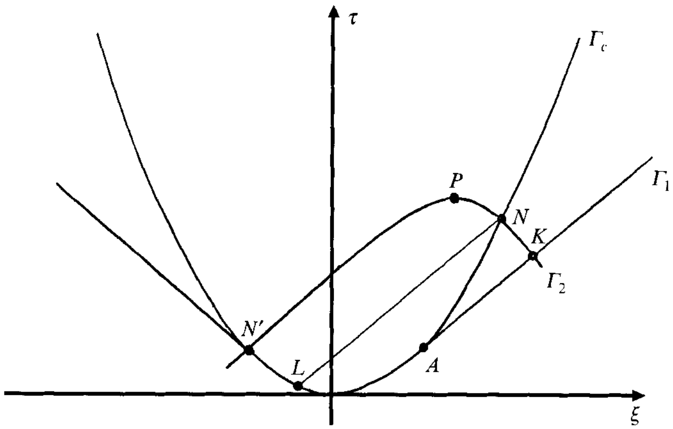

FIGURE 5. The geometry of the integration domain for determining the pressure field inside $D_1(t)$.

sides with respect to t, we get the final formula for the pressure distribution inside $D_1(t)$:

$$p(x,y,t) = \frac{1}{\pi}\int_{a_1}^{\xi_L} \frac{\mathrm{d}\xi}{[(t-f(\xi))^2-(x-\xi)^2-y^2]^{\frac{1}{2}}} + \frac{\surd 2}{\pi} v_c(\tau_2) \frac{\{(a_2-\xi_L)(t-\tau_2+a_2-x)\}^{\frac{1}{2}}}{t-\tau_2+v_c(\tau_2)(a_2-x)}. \quad (6)$$

Here the function $f(x)$ describes the form of the entering body, $v_c(t) = \mathrm{d}a(t)/\mathrm{d}t$ is the velocity of the right-hand contact point, and $a_1(x,y,t)$, $a_2(x,y,t)$ and $\xi_L(x,y,t)$ are the coordinates of the intersection points of curve Γ_c with the curve Γ_2 and the line NL (see figure 5). In (6) let us denote the first term by $p_1(x,y,t)$ (it is always positive) and the second one by $p_2(x,y,t)$.

The expressions obtained show that the pressure changes continuously when crossing the front of the expansion wave l_1. Indeed, on approaching the back side of the front we have $a_2 \to a_* + 0$, $\xi_L \to a_* - 0$, and then $p_1(x,y,t)$ tends to the value of the pressure in front of the expansion wave (see Korobkin 1992*b*), and $p_2(x,y,t) \to 0$. Approaching the free surface where $y \to -0$, $x > a(t)$, one finds that $\xi_L \to a_1$, $t-\tau_2+a_2-x \to 0$ and, hence, $p \to 0$.

Let us denote the time at which the left-hand expansion wave overtakes the right-hand contact point by T_1. The quantity is a solution of the equation $T_1 - a(T_1) = T + a_*$; for a parabolic shape it is $\frac{11}{2}$. Formula (6) allows us to obtain the pressure distribution over the contact spot, where $y = 0$, $|a_* - t + T| < x < a(t)$, $T < t < T_1$, in this region:

$$p(x,0,t) = \frac{1}{\pi}\int_{a_1}^{\xi_L} \frac{\mathrm{d}\xi}{[(t-f(\xi))^2-(x-\xi)^2]^{\frac{1}{2}}} + \frac{2}{\pi}\frac{v_c(\tau_2)}{1+v_c(\tau_2)}\left(\frac{a_2-\xi_L}{a_2-x}\right)^{\frac{1}{2}}.$$

It can be seen that when $x \to a(t) - 0$ the pressure has an integrable singularity. The last expression agrees with that obtained by Korobkin (1985).

Let us consider the pressure distribution behind the front of the expansion wave in detail. It should be noted that if the form of the entering body is described by a smooth function then $p(x,y,t)$ will be a smooth and bounded function inside $D_1(t)$ and, hence, it or its derivatives may have peculiarities only on the boundary of the region or at the moment of the region appearance. Four zones are distinguished; inside each of them the structures of the pressure distribution are complicated and have to be analysed separately. These zones are (i) the vicinities of the contact points at the subsonic stage

$(T < t < T_1)$; (ii) the vicinities of the contact points just after the escape of the shock wave onto the free surface $(t \to T+0)$; (iii) the vicinity of the front of the expansion wave l_1; (iv) the vicinity of the point where the front l_1 is attached to the free surface. Inside each of those zones it is necessary to construct the asymptotics of the pressure and verify their correspondence to the assumptions that lie at the heart of the acoustic approximation.

This analysis corresponds to the method of matched asymptotic expansions which allows us to reduce a complicated problem to consideration of its simplest elements, some effects such as nonlinearity, compressibility and so on being taken into account only within the zones where they are of major importance. On the other hand, the present analysis indicates that it is necessary to develop some special experimental methods to provide information on the fine flow structure inside these zones.

3.1. *The pressure distribution close to the contact point*

It is convenient to use the 'internal' variables x_1, y_1 determined by the equations

$$x = a(t)+x_1, \quad y = y_1, \quad x_1 = \rho\cos\theta_1, \quad y_1 = \rho\sin\theta_1, \tag{7}$$

where $-\pi < \theta < 0$, $\rho \ll 1$. Substitution of (7) in (6) gives, as $\rho \to 0$,

$$p(x,y,t) = \frac{\surd 2}{\pi}\left\{\frac{a(t)-\xi_L}{1+v_c(t)}\right\}^{\frac{1}{2}}\left\{\cos^2\theta_1+\frac{1-v_c^2(t)}{v_c^2(t)}\right\}^{-\frac{1}{2}}$$
$$\times\left[v_c(t)\left\{\cos^2\theta_1+\frac{1-v_c^2(t)}{v_c^2(t)}\right\}^{\frac{1}{2}}-\cos\theta_1\right]^{\frac{1}{2}}\rho^{-\frac{1}{2}}+O(\rho^{\frac{1}{2}}). \tag{8}$$

The formula can be simplified with a suitable stretching of the 'internal' variables. Namely, let us define the new variables x_2, y_2 in the following way:

$$x_1 = [1-v_c^2(t)]^{\frac{1}{2}}x_2, \quad y_1 = y, \quad x_2 = r\cos\theta, \quad y_2 = r\sin\theta. \tag{9}$$

In the new coordinate system the asymptotics of the pressure close to the contact point is

$$p(x,y,t) = \gamma_c(t)\frac{\sin(-\theta/2)}{r^{\frac{1}{2}}}+O(r^{\frac{1}{2}}), \tag{10}$$

$$\gamma_c(t) = \frac{2v_c(t)\,[a(t)-\xi_L]^{\frac{1}{2}}}{\pi[1+v_c(t)]^{\frac{3}{4}}[1-v_c(t)]^{\frac{1}{4}}}. \tag{11}$$

It is worth noting that the function $\gamma_c(t)$ depends not only on local characteristics of the liquid flow, but also on all the previous history of the entry process. Formulae (10), (11) are valid when $T < t < T_1$, but it is obvious that for $t > T_1$ the asymptotic pressure close to the contact point has the same structure; however, the coefficient $\gamma_c(t)$ can be calculated only numerically.

The function $v_c(t)$ is the ratio of the dimensional velocity of the contact point to the sound velocity c_0 and, hence, it is the 'internal' Mach number (see Lesser 1981). Therefore the deformation of the horizontal coordinate in (9) corresponds to the Prandtl transformation (Prandtl 1930) which is used in the linear theory of a thin wing in a steady subsonic flow. This transformation sets the correspondence between characteristics of subsonic flow around a thin wing and characteristics of an ideal, incompressible flow around the body. In our case (10) shows that in the deformed coordinate system (9) with its centre at the right-hand contact point the pressure has

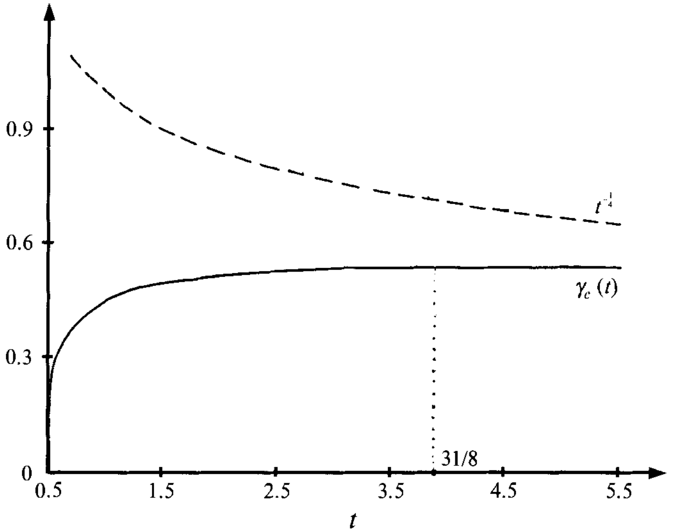

FIGURE 6. Graph of the coefficient $\gamma_c(t)$ of the pressure singularity for a parabolic contour. The function $\gamma_c(t)$ reaches its maximal value at $t = \frac{31}{8}$. The dashed line corresponds to the coefficient in the model for an incompressible liquid.

an integrable singularity of the same kind as that within the framework of the incompressible liquid model (see Korobkin & Pukhnachov 1985). However, the coefficient $\gamma_c(t)$ of the singularity cannot be determined by the liquid flow at the moment t as it occurs for the model of an incompressible liquid.

Thus, close to the contact points at the subsonic stage of the impact the acoustic approximation fails and we need to use some other assumptions on the flow structure inside this narrow zone. The function $\gamma_c(t)$ is very important because it determines the thickness of the spray jet. For a parabolic contour when $\frac{1}{2} < t < \frac{11}{2}$ we get

$$\gamma_c(t) = \frac{(15)^{\frac{1}{2}}}{\pi} \frac{v_c^{\frac{1}{2}}}{1+v_c} [1 - v_c^2]^{\frac{1}{4}}, \quad v_c(t) = \frac{3}{(5+8t)^{\frac{1}{2}}}. \tag{12}$$

Figure 6 presents a plot of the function $\gamma_c(t)$; the dashed line corresponds to the coefficient in the model an incompressible liquid. For a parabolic contour it is $(t)^{-\frac{1}{4}}$. As can be seen, $\gamma_c(t)$ rapidly increases, then it changes rather slowly, reaches its maximal value and starts to decrease. It can be assumed that the function $\gamma_c(t)$ also continues to decrease when $t > \frac{11}{2}$ and approaches asymptotically the function $(t)^{-\frac{1}{4}}$. It follows from (12) that $\gamma_c(t)$ reaches its maximal value, which is $\pi^{-1}(\frac{25}{3})^{\frac{1}{4}}$, at the moment when the velocity of the contact point is exactly equal to half of sound velocity, $v_c = \frac{1}{2}$. This happens when $t = 3.875$.

According to (12) $\gamma_c(t) \to 0$ when $t \to T+0$ and the singularity in (10) disappears. But then $v_c(t) \to 1-0$ and the transformation (9) degenerates. In this case we need to use the general formula (8). Its analysis shows that the pressure asymptotics is not uniformly valid relative to θ_1 when $t \to T+0$. That is why the liquid motion close to the contact points just after the escape of the shock wave onto the free surface has to be analysed separately.

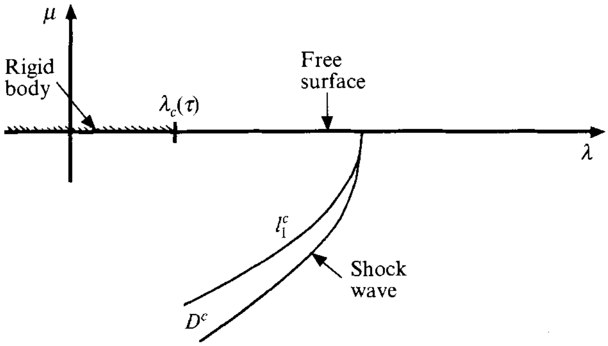

FIGURE 7. The flow pattern near the right-hand contact point just after the shock escapes onto the free surface.

3.2. *The pressure distribution near the contact point at the beginning of the subsonic stage*

In order to find the pressure distribution inside this zone we will use the same moving coordinate system as for the supersonic stage (see Korobkin 1992*b*). Namely, let us introduce 'internal' variables λ, μ, τ such that $x = \mathscr{a}(t)+\lambda$, $y=\mu$, $t = T+\tau$. The function $\mathscr{a}(t)$ describes the motion of the right-hand point of intersection of the entering contour with the initially undisturbed liquid surface ($y = 0$). For all time the equality $f[\mathscr{a}(t)] \equiv t$ is valid, and $\mathscr{a}(t) \equiv a(t)$ when $0 < t < T$, and $\mathscr{a}(t) < a(t)$ when $t > T$.

On the plane of the spatial variables λ, μ the flow region $\mu < 0$ consists of two parts. The first part $D^c(t)$ is bounded below by the shock wave, and above by the front of the expansion wave $l_1^c(t)$ (see figure 7). Inside this region the fluid motion does not depend on the presence of the free surface. Inside the second part, which lies above the curve $l_1^c(t)$, the presence of the free surface is important. The asymptotic pressure inside $D^c(t)$ when $\tau \to -0$ has been derived earlier (see Korobkin 1992*b*) and has the form

$$p(x,y,t) = (-\tau)^{-\frac{1}{2}}\, U(\lambda/\tau^2, \mu/(-\tau)^{\frac{3}{2}}) + \dots .$$

It is natural to expect that the structure of the leading term of the asymptotic pressure will be the same when $\tau \to +0$. Indeed, a simple generalization of the reasoning (see Korobkin 1992*b*) allows us to write the following approximate formula for the pressure distribution inside $D^c(t)$:

$$p(x,y,t) = \frac{1}{\pi}[\tau|a''_-(T)|]^{-\frac{1}{2}} \int_{k_1}^{k_2} \frac{\mathrm{d}k}{\{(k_2-k)(k_3-k)(k-k_1)\}^{\frac{1}{2}}}[1+o(1)].$$

Here $a''_-(T) = a''(T-0)$, and $k_1 < k_2 < 0 < k_3$ are the real roots of the cubic equation

$$|a''_-(T)|\, k^2(2+k) + 2k(\lambda/\tau^2) - \mu^2/\tau^3 = O(\tau). \tag{13}$$

The position of the lower boundary of $D^c(t)$ is determined by the condition of the merging the roots k_1 and k_2. In this case $a_2(x,y,t) = a_* + \tau(1+k_2) + O(\tau^2)$, but inside $D^c(t)$ we have $a_2 < a_*$, therefore $k_2 < -1$. On substituting $k = -1$ in (13), we find the position of the front of the expansion wave for small positive τ:

$$l_1^c(t):\quad \lambda/\tau^2 = -\tfrac{1}{2}(\mu/\tau^{\frac{3}{2}})^2 + \tfrac{1}{2}|a''_-(T)| + O(\tau). \tag{14}$$

Now we shall analyse the pressure distribution above parabola (14). Inside this region $a_2(x,y,t) > a_*$, $T < \tau_2(x,y,t) < t$. The function $\tau_2(x,y,t)$ is to satisfy the equation

$$(t-\tau_2)^2 = (x-a(\tau_2))^2 + y^2, \tag{15}$$

the solution of which we will seek in the form $\tau_2 = T+\tau n$, $n = O(1)$ as $\tau \to 0$. On substitution of this representation in (15), neglecting terms of $O(\tau^4)$ and higher, we obtain

$$(1-n)[a''_-(T)-a''_+(T)n^2+2(\lambda/\tau^2)]+\mu^2/\tau^3 = O(\tau). \tag{16}$$

It can be shown that behind the front (14) this cubic equation has three real roots: $n_1 < 0 < n_2 < 1 < n_3$. But $a_2 - a_* = \tau n + O(\tau^2) > 0$ and $t-\tau_2 = \tau(1-n) > 0$, hence one needs to choose the root $n_2(\lambda/\tau^2, \mu/\tau^{\frac{3}{2}})$.

The asymptotic expansion of $a_1(x,y,t)$ when $t-T \to +0$ has the same form as inside $D^c(t)$, namely $a_1 = \alpha(t)+\tau k_1 + O(\tau^2)$, where $k_1(\lambda/\tau^2, \mu/\tau^{\frac{3}{2}})$ is the smallest root of (13). The function $\tau_L(x,y,t)$ satisfies the equation $\tau_L = \alpha(\tau_L) - a_2 + \tau_2$ which yields

$$\tau_L = T - \tau k_0 n_2 + O(\tau^2), \quad k_0 = [a''_+(T)/a''_-(T)]^{\frac{1}{2}}$$

when $\tau = 0$. For a smooth shape we have $k_0 = \frac{2}{3}$. Substitution of these asymptotic expansions into (6) yields the final relation

$$p(x,y,t) = \frac{1}{\pi\tau^{\frac{1}{2}}}\left\{|a''_-(T)|^{-\frac{1}{2}}\int_{k_1}^{-n_2k_0-1}\frac{\mathrm{d}k}{[(k-k_1)(k_2-k)(k_3-k)]^{\frac{1}{2}}} - \frac{2[n_2(1-n_2)(1+k_0)]^{\frac{1}{2}}}{\mu^2/\tau^3+2|a''_+(T)|\,n_2(1-n_2)^2}\frac{\mu}{\tau^{\frac{3}{2}}}\right\}. \tag{17}$$

When $\mu = 0$ equation (16) has the roots

$$n^{(1)} = 1, \quad n^{(2)} = [k_0^{-2} - 2(\lambda/\tau^2)/|a''_+(T)|]^{\frac{1}{2}}, \quad n^{(3)} = -n^{(2)}.$$

If $\lambda < \lambda_c(\tau) = \tau^2|a''_+(T)|(k_0^{-2}-1)/2$, then one needs to put $n_2 = 1$. When $\lambda = \lambda_c$ the roots $n^{(1)}$ and $n^{(2)}$ merge and for $\lambda > \lambda_c$ we must put $n_2 = n^{(2)}$. In this case $n^{(2)} < 1$. The function $\lambda_c(\tau)$ gives the position of the contact point in the moving coordinate system, more exactly $a(t) - \alpha(t) = \lambda_c(\tau) + O(\tau^3)$. Equation (13) gives $k_1 = -1-k_0 n^{(2)}$, $k_2 = 0$, $k_3 = -2-k_1$ when $\mu = 0$. On the surface of the entering body ($\mu \to -0, \lambda < \lambda_c$) one has $n_2 = 1$, and (16) yields $\mu^2/[\tau^3(1-n_2)] = 2(\lambda_c - \lambda)/\tau^2$. Then

$$p(x,0,t) = \frac{1}{\pi}\left\{|a''_-(T)|\int_1^{n^{(2)}}\frac{\mathrm{d}\sigma}{[(n^{(2)}-\sigma)(n^{(2)}+\sigma)(1+\sigma k_0)]^{\frac{1}{2}}} + \left(\frac{2(1+k_0)\tau^2}{\lambda_c-\lambda}\right)^{\frac{1}{2}}\right\}\tau^{-\frac{1}{2}} + \dots.$$

When $\lambda \to \lambda_c - 0$ the first term tends to zero and the second one to infinity. On the free surface ($\mu = 0, \lambda > \lambda_c$) the following equations hold: $n_2 < 1$, $k_1 = -1-k_0 n_2$, hence both terms in (17) are equal to zero.

Hence, in the vicinity under consideration, like when $\tau < 0$, the pressure field is described with the help of the self-similar variables λ/τ^2, $\mu/\tau^{\frac{3}{2}}$. But now the free surface appears in the flow scheme (see figure 7), and inside the regions which lie above and below the parabola l_1^c the pressure is given by two different expressions. In the special case when $a''_-(T) = 0$ the present analysis fails and it is necessary to carry out additional investigations.

3.3. *The pressure field near the expansion wave front*

We shall determine the asymptotic expansion of the pressure $p(x,y,t)$ near the front of the expansion wave l_1. Inside the vicinity under consideration $x = a_* + (t-T-\Delta)\cos\theta$, $y = (t-T-\Delta)\sin\theta$, where $0 < \Delta \ll 1$, $-\pi \leqslant \theta \leqslant 0$ when $T < t < T+a_*$ and $-\arccos(a_*/(t-T)) \leqslant \theta \leqslant 0$ when $t > T+a_*$ (see figure 1*b*). The main difficulties arising in the analysis of (6) for $\Delta \to 0$ are connected with the construction of the asymptotic expansion of the integral summand $p_1(x,y,t)$. The reason is that both the integration limits and the integrand depend on the small parameter Δ; moreover the integrand is an unbounded function and the positions of its singularities change with Δ.

The pressure variation Δp on the front of the expansion wave can be shown to be

$$\left.\begin{aligned} \Delta p &= -\Delta^{\frac{3}{2}} f''(a_*) f_c(k_0)(t-T)^{-\frac{1}{2}}(1-\cos\theta)^{-3} + O(\Delta^{\frac{5}{2}}), \\ f_c(k_0) &= \frac{\sqrt{2}}{3\pi}(1+k_0)^{\frac{1}{2}}(3k_0^2+k_0-2). \end{aligned}\right\} \tag{18}$$

For a parabolic body form $f''(a_*) = 1$, $k_0 = \frac{2}{3}$ (see Korobkin 1992*a*), but $f_c(\frac{2}{3}) = 0$, and, hence, the pressure variation provided by the presence of the free surface has order not lower than $O(\Delta^{\frac{5}{2}})$. This result is also valid for an arbitrary smooth contour with the only restriction being $f''(a_*) \neq 0$.

Thus, when the Mach number M is small, the acoustic approximation describes the geometry of the front of the expansion wave and the liquid flow behind it correctly except for a narrow zone near the point where this front joins the free surface. To construct the leading-order asymptotics of the function Δp when $\Delta \to +0$ it is necessary to consider the high-order terms, which lead to an increase in the number of calculations. On the other hand, the exact form of the leading term is less important than the exact order of Δp for small Δ. It was shown in this Section that the order of Δp is not lower than $O(\Delta^{\frac{5}{2}})$ but it can be higher. An improvement of this result will be given in the following Section by an indirect method, without construction of the asymptotic expansion of $\Delta p(x, y, t)$ in the explicit form. In any case we can say that the pressure behind the expansion waves under consideration drops more slowly than in the general case.

3.4. *The asymptotic pressure near the point where the front of the expansion wave is attached to the free surface*

As has been mentioned earlier (see Korobkin 1992*b*), the high order of the touching of the shock front and the expansion wave front close to the point of their attachment to the free surface can lead to unbounded values of the pressure gradient in this vicinity. Using (6) it can be shown that this is in fact so. For this purpose it is convenient to use the coordinate system Δ, θ introduced in §3.3, associated with the front of the expansion wave. The small vicinity considered now is determined by the relations $0 < \Delta \ll 1$, $0 < -\theta \ll 1$, where the respective orders of the values Δ and θ are unknown in advance and must be determined together with the construction of the asymptotic pressure $p(x, y, t)$ when $\Delta \to +1$, $\theta \to -0$. Instead of θ it is convenient to introduce the new variable $\epsilon = 1 - \cos\theta$.

Construct first the asymptotic expansion of the function $\tau_2(x, y, t)$ when $\epsilon \to 0$, $\Delta \to 0$. Equation (15) gives

$$\left.\begin{aligned} (t-\tau_2)^2 &= (t-T-\Delta)^2 - 2(a_2-a_*)(t-T-\Delta)(1-\epsilon) + (a_2-a_*)^2, \\ a_2 &= a(T+[\tau_2-T]). \end{aligned}\right\} \tag{19}$$

It is clear that $\tau_2 - T \to 0$, $a_2 - a_* \to 0$ when $\Delta \to 0$, and

$$a(\tau_2) = a_* + \tau_2 - T + \tfrac{1}{2}a''_+(T)(\tau_2 - T)^2 + \dots.$$

Taking the last expansion into account we get from (19) a single equation for τ_2,

$$|a''_+(T)|(\tau_2 - T)^2 + 2\epsilon(\tau_2 - T) - 2\Delta = o(1),$$

where the right-hand side contains terms of order higher than that of at least one of

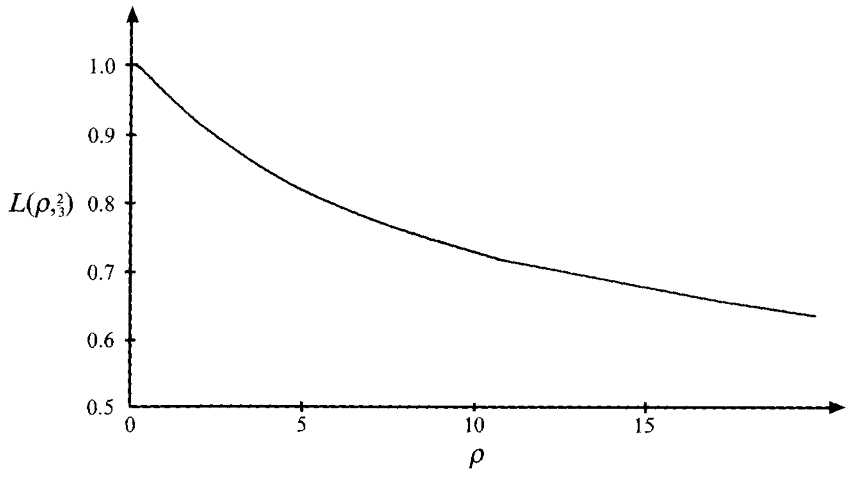

FIGURE 8. Graph of the function $L(\rho, \frac{2}{3})$.

the terms on the left-hand side of the equation. A non-trivial asymptotic expression for the function $\tau_2(x, y, t)$ will occur only when all three terms on the left-hand side of the equation have the same order. Thus, we need to put $\Delta = \rho\epsilon^2/(2|a''_+(T)|)$, where ρ is the new self-similar variable and $\rho = O(1)$ at $\epsilon \to 0$.

The final formula for the pressure is

$$\left.\begin{aligned} p(x,y,t) &= p_{sw}(a_*, t)\, L(\rho, k_0) + O(\epsilon), \\ p_{sw}(a_*, t) &= [f''(a_*)(t-T)]^{-\frac{1}{4}}, \end{aligned}\right\} \tag{20}$$

where $L(\rho, k_0) = 1$ when $-k_0^2 < \rho < 0$, and

$$L(\rho, k_0) = \frac{1}{2} + \frac{[2(1+k_0)]^{\frac{1}{2}}}{\pi k_0}\left\{\frac{(1+\rho)^{\frac{1}{2}}-1}{1+\rho}\right\}^{\frac{1}{2}} + \frac{1}{\pi}\arcsin\left[\frac{1+k_0-(1+\rho)^{\frac{1}{2}}}{(\rho+k_0^2)^{\frac{1}{2}}}\right]$$

when $\rho > 0$; $p_{sw}(a_*, t)$ is the pressure on the back side of the shock front at the point of its attachment to the free surface. Thus, the pressure is constant and is equal to $p_{sw}(a_*, t)$ between the shock wave and the front of the expansion wave $(-k_0^2 < \rho < 0)$. The pressure distribution is described by the function $L(\rho, k_0)$ which does not change with time between the front of the expansion wave and the free surface $(\rho > 0)$. The pressure magnitude is proportional to $p_{sw}(a_*, t)$ and vanishes with time as $t^{-\frac{1}{4}}$. Along the curves $\rho = \text{const.}$, i.e.

$$\Delta = 0.5\rho(1-\cos\theta)^2/|a''_+|, \tag{21}$$

the pressure is constant. The function $L(\rho, k_0)$ tends to zero and to unity approaching the free surface $(\rho \to \infty)$ and the front of the expansion wave $(\rho \to +0)$, respectively. The structure of the pressure distribution in this vicinity is close to that in the Prandtl–Meyer flow. A graph of the function $L(\rho, k_0)$ at $k_0 = \frac{2}{3}$, corresponding to the smooth-body case, is shown in figure 8.

The vicinity considered is a narrow zone attached to the front of the expansion wave. In fact, if the size of the zone along the front is small and equal to ϵ_0, then the zone width is of $O(\epsilon_0^4)$, which follows from (21). Correspondingly, in this zone the normal to the front of the expansion wave derivative of the pressure is equal to

$$\frac{\partial p}{\partial \Delta} = p_{sw}(a_*, t)\, L'_\rho(\rho, k_0)\frac{2|a''_+|}{\epsilon^2}$$

and is of $O(1/\Delta)$ as $\Delta \to +0$. Hence, the assumption that the pressure gradient is finite over the flow region, which lies at the heart of the acoustic approximation, breaks down in the above-mentioned zone. Inside this zone the acoustic approximation fails and we must construct an 'internal' asymptotic expansion of the solution which is able to describe the fine structure of the flow near the points where the shock wave attaches to the free surface.

On going away from the free surface along the front of the expansion wave the asymptotics (20) has to be matched with that of (18) in the region where they are simultaneously valid. But for finite values of θ and small Δ we have $\rho \ll 1$ and $\rho = O(\Delta)$ when $\Delta \to 0$. Therefore, when $\Delta \to 0$ the real order of the pressure variation Δp in crossing the front of the expansion wave has to coincide with the order of the function $L(\rho, k_0) - 1$ at $\rho \to 0$. Analysis of the last function for small positive ρ is not difficult and yields

$$L(\rho, k_0) - 1 = \frac{[2(1+k_0)]^{\frac{1}{2}}}{\pi k_0}\left[-\frac{1}{3k_0^2}[3k_0^2 + k_0 - 2](\rho/2)^{\frac{3}{2}} + \frac{1}{20k_0^4}[35k_0^4 + 5k_0^3 - 24k_0^2 - 28k_0 - 34](\rho/2)^{\frac{5}{2}} + O(\rho^{\frac{7}{2}})\right].$$

For a smooth contour ($k_0 = \frac{2}{3}$) we get

$$L(\rho, \tfrac{2}{3}) - 1 = -\frac{1335}{256\pi}\rho^{\frac{5}{2}} + O(\rho^{\frac{7}{2}}).$$

Therefore, near the front of the expansion wave the pressure $p(x, y, t)$ is a twice continuously differentiable function and its third derivative normal to the front is of $O(\Delta^{-\frac{1}{2}})$ as $\Delta \to 0$.

4. The pressure field inside the region $D_2(t)$

The region of the expansion wave interaction, $D_2(t)$, appears at the moment at which the left-hand (l_1) and right-hand (l_1') fronts first touch. The region is adjacent to the region $D_1(t)$ and is bounded above by the surface of the entering body when $a_* + T < t < T_1$ (see figure 1*b*) and by the fronts of the compression waves l_2 and l_2' when $t > T_1$ (figure 4). If the entering shape is unsymmetrical, then the geometry of this region is more complicated. Let the point U with coordinates x, y lie inside $D_2(t)$. Then the following chain of inequalities is valid:

$$-a^* < a_1 < -a_* < a_* < a_2 < a^*,$$

where $a^* = a(T_1)$. Moreover $a_1 = -a^*$ when $U \in l_2'$ and $a_2 = a^*$ when $U \in l_2$.

Inside the region the velocity potential $\phi(x, y, t)$ is determined by (2), where the integration domain is shown in figure 9. This figure differs from figure 2 by the presence of lines NL and $N'L'$ which have the characteristic slope and pass through the points N and N', respectively. With the help of these lines the integration domain $\sigma(x, y, t)$ is decomposed into six parts: $NBN'P$, $NAL'B$, $N'BLA'$, NKA, $N'K'A'$, LBL'. In §3 it was pointed out that the integrals in (2) over the domains $NAL'B \cup LBL'$ and NKA, $N'BLA' \cup LBL'$ and $N'K'A'$, differ from each other only in their sign. Therefore the integration in (2) occurs in fact over two domains $NBN'P$ and LBL' and the integral

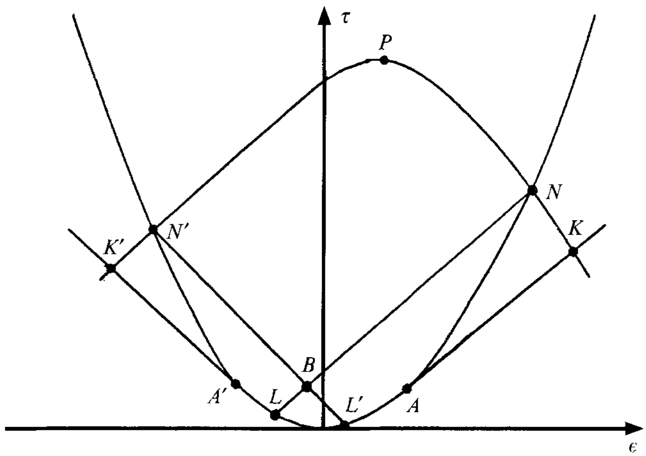

FIGURE 9. Geometry of the integration domain for determining the pressure field inside $D_2(t)$.

over LBL' has to be taken with a minus sign (see Krasilshchikiva 1982). It is important that in the final formula the integration is carried out over the domains where $\phi_y(\xi, 0, \tau) \equiv -1$. This case allows us to study the pressure field inside $D_2(t)$ in detail.

Let us denote the coordinates of the points L and L' by (ξ_L, τ_L) and (ξ'_L, τ'_L), respectively. Then $\tau'_L = f(\xi'_L)$, $\tau_L = f(\xi_L)$ and $-a_* < \xi_L < \xi'_L < a_*$. The pressure distribution inside $D_2(t)$ can be found in the same manner as in the previous section:

$$p(x,y,t) = \frac{\surd 2}{\pi}\left\{ v_c(\tau_1)\frac{[(t-\tau_1-a_1+x)(\xi'_L-a_1)]^{\frac{1}{2}}}{t-\tau_1+v_c(\tau_1)(x-a_1)} + v_c(\tau_2)\frac{[(t-\tau_2+a_2-x)(a_2-\xi_L)]^{\frac{1}{2}}}{t-\tau_2+v_c(\tau_2)(a_2-x)}\right\} - \frac{1}{\pi}\int_{\xi_L}^{\xi'_L}\frac{d\xi}{[(t-f(\xi))^2-(x-\xi)^2-y^2]^{\frac{1}{2}}}. \tag{22}$$

This formula is also valid in the case when the segments NL and $N'L'$ do not intersect. Then the domain LBL' is absent but the integration domain $NLL'N'P$ may be added up to the rectangular $NLBL'N'P$ by continuing the lines NL and $N'L'$ up to their intersection at the point B. Putting formally $\phi_y(\xi, 0, \tau) \equiv -1$ in LBL', adding and subtracting in (2) the corresponding integral over the additional domain LBL', we will arrive at the same equation (22), but now $\xi'_L < \xi_L$ and the integral term in (22) will be negative.

For an unsymmetrical contour the laws of motion of the left-hand and the right-hand contact points are different; however, (22) remains valid, but noting that in the first term, $v_c(\tau_1)$ is the velocity of the left-hand contact point at the moment τ_1 and in the second term, $v_c(\tau_2)$ is the velocity of the right-hand contact point at the moment τ_2.

It should be noted that for an arbitrary contour, at every time moment we can indicate inside the flow region a curve below which the pressure is positive. The curve is given by the equation $\xi'_L(x,y,t) = \xi_L(x,y,t)$. Above this curve $\xi'_L < \xi_L$, therefore, the integral in (22) is positive and it is possible that the pressure will be negative. The possibility of this event depends on the magnitude of the velocity of the contact points at the subsonic stage. More exactly, this velocity must be less than some critical value.

To clarify this statement, let us consider the entry of a rectangular cylinder, with a curved base of radius of curvature R. Then the supersonic stage can be found and analysed as earlier, but at the subsonic stage one has $v_c(t) = 0$ when $t > T_R > T$, where T_R depends on the radius of curvature of the base and $T_R < T_1$. In this case the first two terms in (22) are zero and, hence, the pressure is negative.

The analysis of both the pressure distribution (22) and the entering body forms $f(x)$ leading to the formation of rarefaction zones is impossible without numerical calculations. It should be noted that the numerical integration in (2) should present no problems, since the integrand is a bounded and smooth function in the integration interval.

In conclusion we would like to give the expression for the pressure at the top of the entering symmetrical body. Equation (22) when $x = y = 0$ and $t > T + a_*$ gives

$$p(0,0,t) = \frac{4}{\pi}\frac{v_c(\tau_2)}{1+v_c(\tau_2)}\left[1-\frac{\xi_L}{a_2}\right]^{\frac{1}{2}} - \frac{2}{\pi}\int_0^{-\xi_L}\frac{\mathrm{d}\xi}{[(t-f(\xi))^2-\xi^2]^{\frac{1}{2}}}. \tag{23}$$

When $t \to T + a_* + 0$ we have $\tau_2 \to T+0$, $a_2 \to a_* + 0$, $\xi_L \to a_* - 0$, so the first term in (23) tends to zero and the second one to the pressure at the top of the entering contour just before the expansion waves reach this point (see Korobkin 1992*b*).

5. The pressure distribution over the wetted part of the entering parabolic contour

For a parabolic contour the formulae obtained above can be simplified and written with the help of special functions. Indeed, in this case one has $f(\xi) = \xi^2/2$ and, therefore, the expression $(t-f(\xi))^2 - (x-\xi)^2 - y^2$ in (6) and (22) is a polynomial of the fourth degree with respect to ξ. The polynomial has four real roots $a_0 < a_1 < a_2 < a_3$ (see Korobkin 1992*b*) which are the coordinates of the intersection points of the curves $\Gamma_0 : \tau = f(\xi)$ and Γ_2 (roots a_1, a_2), and the curves Γ_0 and Γ_3, which is the mirror image of Γ_2 relative to the line $\tau = t$ (roots a_0, a_3). Hence, the integral terms in (6) and (22) can be expressed with the help of elliptic integrals of the first kind. For (6) the following chain of the inequalities is valid: $a_0 < -a_* < a_1 < \xi < \xi_L < a_* < a_2 < a_3$, therefore

$$\int_{a_1}^{\xi_L}\frac{\mathrm{d}\xi}{[(t-f(\xi))^2-(x-\xi)^2-y^2]^{\frac{1}{2}}} = \frac{4F[\varphi(\xi_L), q]}{[(a_3-a_1)(a_2-a_0)]^{\frac{1}{2}}}, \tag{24}$$

$$\varphi(z) = \arcsin\left[\left(\frac{(a_2-a_0)(z-a_1)}{(a_2-a_1)(z-a_0)}\right)^{\frac{1}{2}}\right], \quad q = \left(\frac{(a_2-a_1)(a_3-a_0)}{(a_3-a_1)(a_2-a_0)}\right)^{\frac{1}{2}}.$$

Correspondingly, for (22) independently of the relative position of the points L and L' we obtain

$$\int_{\xi_L}^{\xi'_L}\frac{\mathrm{d}\xi}{[(t-f(\xi))^2-(x-\xi)^2-y^2]^{\frac{1}{2}}} = \frac{4\{F[\varphi(\xi'_L), q]-F[\varphi(\xi_L), q]\}}{[(a_3-a_1)(a_2-a_0)]^{\frac{1}{2}}}. \tag{25}$$

Taking (24) and (25) into account one can divide the procedure for evaluating the pressure field inside the regions D, D_1, D'_1, D_2 into two steps. As the first step the coordinates of the intersection points of the curves Γ_0 and Γ_2, and Γ_0 and Γ_3, and also the curves Γ_c and Γ_2 are determinated. This can be done graphically or by solving the algebraic equations. Then the pressure is evaluated using the formulae (6), (22), (24), (25). In view of the contour symmetry it is enough to consider only the region $x > 0$.

For the contact spot ($y = 0, |x| < a(t)$) the above algebraic equations can be solved

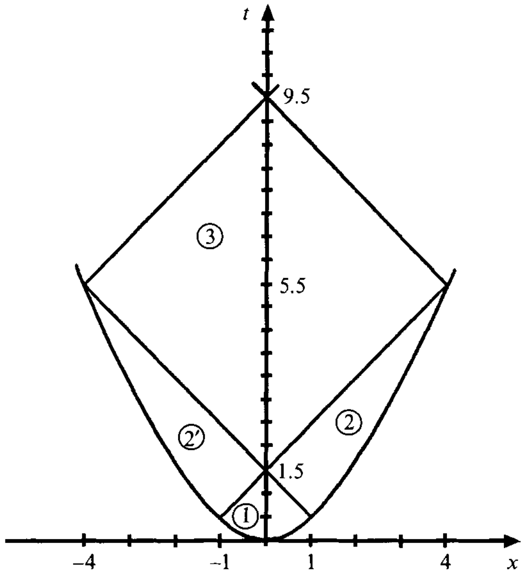

FIGURE 10. Geometry of the domains on the plane (x, t) for which the pressure distribution over the wetted part of the entering contour is analytically given in the present paper.

analytically. The equation for α_j $(j = 0, 1, 2, 3)$ when $y = 0$ splits into two quadratic equations. When $|x| < a(t)$ this allows us to write the following relations:

$$\alpha_0 = -1-[2(t+x)+1]^{\frac{1}{2}}, \quad \alpha_1 = 1-[2(t-x)+1]^{\frac{1}{2}},$$
$$\alpha_2 = -1+[2(t+x)+1]^{\frac{1}{2}}, \quad \alpha_3 = 1+[2(t-x)+1]^{\frac{1}{2}}.$$

When $\alpha(t) < x < a(t)$ the equations for α_2 and α_3 change places. Denote the expression $[(t+\frac{1}{2})^2-x^2]^{\frac{1}{2}}$ by $h(x,t)$ and the expression $2[(\alpha_3-\alpha_1)(\alpha_2-\alpha_0)]^{-\frac{1}{2}}$ by $W(x,t)$ then we have

$$W(x,t) = (2h)^{-\frac{1}{2}}, \quad q(x,0,t) = \left[\frac{1}{2}+\frac{t-\frac{1}{2}}{2h}\right]^{\frac{1}{2}}$$

when $0 < x < \alpha(t)$,

$$W(x,t) = (t-\tfrac{1}{2}+h)^{-\frac{1}{2}}, \quad q(x,0,t) = \left[\frac{1}{2}+\frac{t-\frac{1}{2}}{2h}\right]^{-\frac{1}{2}}$$

when $\alpha(t) < x < a(t)$. For other expressions we get

$$E_\pm = [6+2(t\pm x)]^{\frac{1}{2}}, \quad \xi_L = 4-E_+, \quad \xi_L' = E_- - 4, \quad \tau_1 = \tfrac{1}{2}[1+E_-^2-3E_-],$$
$$\tau_2 = \tfrac{1}{2}[1+E_+^2-3E_+], \quad a_1 = \tfrac{1}{2}[7-3E_-], \quad a_2 = \tfrac{1}{2}[3E_+-7],$$
$$v_c(\tau_1) = 3/(2E_- - 3), \quad v_c(\tau_2) = 3/(2E_+ - 3).$$

It can be seen that $\tau_1(x,0,t) = \tau_2(-x,0,y)$, $\xi_L(x,0,t) = -\xi_{L'}(-x,0,t)$. The formulae for $\varphi(\xi_L)$ and $\varphi(\xi_L')$ cannot be simplified, therefore the final formulae for the pressure distribution over the wetted part of the entering contour (see figure 10) are the following:

$$p(x,0,t) = \frac{2}{\pi} W(x,t)\, F[\varphi(\xi_L), q] + S(x,t), \tag{26}$$

$$S(x,t) = \frac{3(5)^{\frac{1}{2}}}{\pi E_+}\left[\frac{E_+-3}{3(E_+-3)+2(1-x)}\right]^{\frac{1}{2}},$$

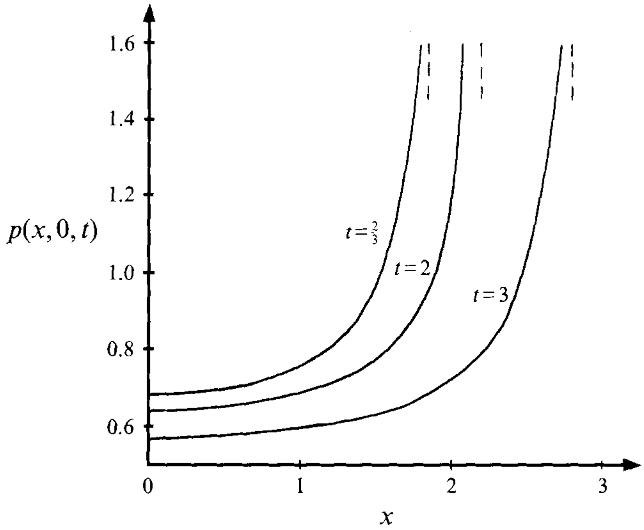

FIGURE 11. Pressure on the contact spot for different times.

when $\frac{1}{2} < t < \frac{11}{2}$, $|\frac{3}{2} - t| < |x| < a(t)$ (domains 2 and 2′);

$$p(x, 0, t) = \frac{2}{\pi} W(x, t) \{F[\varphi(\xi_L), q] - F[\varphi(\xi'_L), q]\} + S(x, t) + S(-x, t), \tag{27}$$

when $\frac{3}{2} < t < \frac{11}{2}$, $|x| < t - \frac{3}{2}$ and $\frac{11}{2} < t < \frac{19}{2}$, $|x| < \frac{19}{2} - t$ (domain 3). It is necessary to add to (26) and (27) the formula given by Rochester (1979), which in our notation has the form

$$p(x, 0, t) = \frac{2}{\pi} W(x, t) F[\tfrac{1}{2}\pi, q] \tag{28}$$

and gives the pressure distribution in domain 1, i.e. when $0 < t < \frac{1}{2}$, $|x| < a(t)$ and $\frac{1}{2} < t < \frac{3}{2}$, $|x| < \frac{3}{2} - t$. The pressure distribution over the contact spot for different times is shown in figure 11. It can be seen that the pressure profiles are smooth, which is connected with smoothness of the pressure near the expansion wave fronts (see §3.3). It may be expected that the compression waves which are initiated by the reflection of the expansion waves in the free surface do not reduce the smoothness of the pressure distribution inside the contact spot.

In order to evaluate the pressure at the top of the parabola ($x = y = 0$) it is convenient to use (23), which gives that the pressure before the expansion waves reach the centre of the contact spot ($0 < t < \frac{3}{2}$) is

$$p(0, 0, t) = \frac{2}{\pi\beta} F\left[\tfrac{1}{2}\pi, \frac{(2t)^{\frac{1}{2}}}{\beta}\right], \tag{29}$$

where $\beta = [2t + 1]^{\frac{1}{2}}$. When $t > \frac{3}{2}$ the pressure depends on the presence of the free surface of the liquid and before the compression waves l_2, l'_2 reach the top it is equal to

$$p(0, 0, t) = \frac{6(5)^{\frac{1}{2}}}{\pi\alpha}\left[\frac{\alpha - 3}{3\alpha - 7}\right]^{\frac{1}{2}} + \frac{4}{\pi(1 + \beta)} F\left[\arcsin\left(\frac{4 - \alpha}{\beta - 1}\right), \frac{\beta - 1}{\beta + 1}\right], \tag{30}$$

where $\alpha = [6 + 2t]^{\frac{1}{2}}$. It should be noted that equation (29) defines the pressure evolution

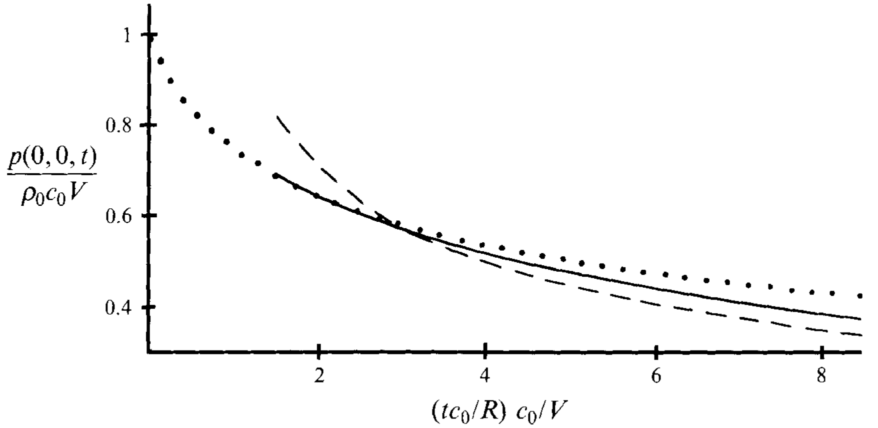

FIGURE 12. Pressure at the top of the entering parabolic contour. ——, exact solution; · · · ·, solution of the impact problem without considering the presence of the free surface; – – – –, Wagner solution for the incompressible liquid model.

at the top of the contour for all time if the free surface is absent, which is equivalent to replacing the boundary condition in (1) on the parts $y = 0$, $|x| > a(t)$ by the slip condition $\phi_y = 0$.

The pressure at the contour top without taking the compressibility of liquid into account was given by Wagner (1932). His formula in our dimensionless variables has the simple form

$$p_W(0,0,t) = \frac{\mathrm{d}c(t)}{\mathrm{d}t},$$

where $c(t)$ is half the contact spot width. For the parabolic contour we have $c(t) = 2t^{\frac{1}{2}}$, then

$$p_W(0,0,t) = t^{-\frac{1}{2}}. \tag{31}$$

The functions (29), (30), (31) are shown in figure 12. The solid line corresponds to the exact solution (30), the dotted line to the solution without considering the presence of the free surface (29), the dashed line to the Wagner solution (31). It is worth noting that the Wagner solution deviates from the precise one at $t = \frac{19}{2}$ by not more than 10% of the latter.

7. Conclusion

In the present paper it is shown that within the framework of the acoustic approximation the pressure distribution both inside the liquid and at the contact spot at the beginning of the subsonic stage of the vertical entry of a solid-body is given in quadratures and for a parabolic contour it is given in explicit form. Analysis of the solution allows us to distinguish zones where the acoustic approximation fails and the liquid flow becomes essentially nonlinear. These are the vicinities of the contact points, and the narrow zones close to the points where the expansion wave is attached to the free surface. The asymptotics of the pressure obtained makes it possible to estimate the relative sizes of the zones and point out the rules of introduction of the 'internal' variables and the new unknown functions which can help us to describe the fine structure both of the liquid flow and the pressure field inside these zones. It is possible that the exact forms of the asymptotic formulae are not very interesting but they will be helpful for matching the 'internal' and 'external' solutions. Outside those zones the pressure field is correctly described by the acoustic theory and the formulae obtained can be used in practice.

Characteristics of the pressure field for large time are of undoubted importance. Such calculations can only be carried out numerically. The number of calculations rapidly increases in time, which leads to reduction of the solution correctness. However, we can hope that the solution of the original problem (1) will be sufficiently close to the solution of the Wagner problem, which does not consider compressibility of liquid, for moderate times. It is worth noting that the calculations can be done with a pocket calculator, as the most expensive part of the work has been carried out analytically in the present paper.

The axisymmetrical problem of the vertical entry of a body of rotation into an ideal slightly compressible liquid can be treated in the same way as the present paper. However, if for the plane problem the geometry of the liquid flow region is given in quadratures, in the axisymmetrical case it is necessary to solve an integral equation.

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